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THE TEACHING OF GEOMETRY

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The Teaching of Geometry

INTRODUCTION

It would be the part of extreme hardihood for anyone today to challenge the statement that the structural and material fabric of the modern world is ordered upon a scientific foundation. Whether the art of peace or the science of war be under consideration, the conclusion must be that of Voss: "Our entire present civilization, as far as it depends upon the intellectual penetration and utilization of nature, has its real foundation in the mathematical sciences."

During the period of this country's participation in the World War, it fell to my lot to teach the theoretical principles of gunnery for heavy artillery; and to every young American who passed through this drastic course of training, it was vividly evident that the principle which directed the training of every great gun upon the German fortifications, of every anti-aircraft gun against the aeroplanes of the Boche, was not national animosity—but the most advanced theory of modern science.

When the shells from the forest of St. Gobain began to fall with regular intermissions upon the city of Paris, even scientists stood aghast with incredulity. It was unbelievable that any gun ever invented could fire a shell a distance of seventy miles. When it became clear that these shells were not dropped from high-flying invisible aircraft but were actually fired from some monster gun concealed somewhere within a radius of seventy miles of Paris, the problem first of locating and then of destroying this death-dealing monster became one of pressing military necessity. By the device of stretching awnings of canvas, of marking the point of entrance, the point of contact with the ground, and calculating the velocity from the distance and time elements, the mathematicians of France were soon able to calculate the direction, range, and trajectory of the invisible monster; and actually to locate its place of concealment in the Forêt de Saint Gobain. Already this marvel is out-moded. Today we hear, almost with complacency, of tests of super-long range guns, which are to fire a projectile a distance of two hundred miles!

Perhaps the Great War, for all the remarkable evidences of scientific skill and ingenuity which it evoked, afforded no more spectacular popular evidence of the efficacy of mathematical science in coping with a terrible, invisible, and almost unknown danger. Geometry, a most powerful weapon in the hands of the trained scientist, thus became the direct and actual means of defending one of the greatest cities in the world from an attack which, at its first outbreak, seemed marked by an atmosphere of almost diabolic mystery and horror.

I. THE AIMS AND RESULTS OF GEOMETRICAL STUDY

Such illustrations, emerging so conspicuously out of the stern experiences of a world war, may suffice to bring home to all of us, in times of peace, the real value of the study of mathematics as a weapon both of offense and defense. As the boy and the girl, in the high school of today, prepares the lesson in geometry for the daily recitation, some renewed consciousness must arise of the value and the utility of such study. Surely the teacher should bring home to the pupil, in as vivid a way as possible, the why and the wherefore of the study of such a subject—its indispensability to the modern world, its usefulness as an instrumentality in the general progress of civilization.

It is not enough, however, to stop at this point in our demonstration of the value of the study of geometry The teacher should endeavor at the very outset to make abundantly clear the value of the study of mathematics to the individual. I would lay down as a fundamental truth the proposition that mathematical study, and in especial geometry, because of its dual appeal—at once to the eye by reason of its pictorial character and to the mind by reason of the relationships involved—exercises in transcendent degree the reasoning power of the individual. To go through a text on plane and solid geometry, with intelligence and judgment, results in the acquisition of an extraordinary range and mass of actual facts regarding the visible universe and the properties and relationships of familiar and chosen objects, forms, and figures which exist in the universe. But the true disciplinary value of the study of geometry arises, not from the extent or volume of facts thus learned—valuable as such knowledge may be. The real object of such study is to develop in the individual the power to utilize the principles employed in this science, to conquer new domains—to solve new problems, to master the recalcitrant difficulties of hidden relationships.

The fundamentals of geometry are easily grasped, since they are basically simple. In the early stages of the subject, the data are few and elemental; and the steps in the reasoning follow with appealing force and singular elegance. Moreover, there is a peculiar satisfaction, which arises from the consciousness that

the results arrived at through legitimate processes of reasoning are not merely certain and true for a particular case, but are universally valid. Moreover, there is a close analogy between the methods of reasoning employed in geometry and the methods of reasoning we employ every day in solving the problems of business affairs, in general, of life itself. Whatever be the problem, one must make clear to himself first of all the exact nature and conditions of the problem; then analyze the various methods by which the difficulty may be resolved; and finally, after eliminating the manifestly unsuitable or ineffective, logically follow out the consequences of the course of action adopted as best suited to the solution of the question. To understand the given situation; to weigh the efficacy of the various means at one's disposal; and to pursue the course chosen with unfaltering logic—these are the phases of the problem, whether one be mathematician or diplomat, engineer or captain of industry.

Not the least of the advantages of the study of geometry, is the stimulant it affords to the latent inventive or constructive faculty of the pupil. If one is given an original exercise to solve, the process of arriving at a solution is a matter of sheer originality—with the individual student—whether or not this same problem has already been solved by a thousand students before him. It is often the case that the mere statement of the problem, accompanied by the drawing of an accurate figure, does not immediately suggest any particular avenue of approach to the solution. In this case, the constructive imagination of the student is at once challenged. If the pupil—after the fashion of the chess player—is able to think several moves ahead, he may be able to see the points and lines of the figures in new relationships, inherent in the figure but not lying upon its very face; and often the drawing of additional lines, the construction of auxiliary figures, will instantly unveil the fundamental relationships sought.

It is when this stage of development is attained by the student that he first appreciates, in all its fulness, the beauty of geometry; and is inspired to make investigations on his own account, independently of the actual assignments of work to be accomplished. This is really the beginning, which is also the end, of all mathematical instruction: the study of the subject for its own sake.

The true purpose of instruction in geometry is to develop the faculty of independent thinking in the field of geometry. The highest type of student is the investigator—he or she who sets out to discover things for himself or herself. It is on that account that, in all my teaching, I place first the development of mental self-reliance. Memory, of necessity, plays a rôle of essential importance in geometry; for one cannot become an independent investigator until he masters and has at immediate mental command—at the fingers' ends of his mind, so to speak—a considerable body of knowledge. But the acquisition of this knowledge is not the end of his study; it should be its beginning. For it is with the equipment of the knowledge thus acquired that he is enabled, through the investigating and originating faculties of his mind, to conquer the problems which arise upon his way—the "originals," as they are happily called, since they come to him with original freshness and original difficulty.

I maintain then, as a sound pedagogical principle—and it is the principle upon which I act in all my teaching—that every pupil, however dull or unresponsive, should be thought of as, potentially, an investigator. And the prime object in teaching the pupil, especially in teaching him such a subject as geometry, is to develop in him, so far as may be possible, within the limits of his faculty and his capability of grasping the subject, the power of independent thinking, of mental self-reliance, of the intellectual courage to tackle with no small show of confidence new problems, difficulties not before encountered, resting in the faith at once of his knowledge and the power of research which his training in the acquisition of that knowledge has developed in him.

In the monograph, "The Functions of Geometry as a Subject of Education," Carson has given a clear and forceful presentation of the advantages which accrue to the individual from the study of geometry:

"The statement that a given individual has received a sound geometrical training implies that he has segregated from the whole of his sense impressions a certain set of these impressions, that he has then eliminated from their consideration all irrelevant impressions (in other words, acquired a subjective command of these impressions), that he has developed on the basis of these

impressions an ordered and continuous system of logical deduction, and finally that he is capable of expressing the nature of these impressions and his deductions therefrom in terms simple and free from ambiguity. . . . The same sequence of mental processes underlies the whole career of any individual in any walk of life if only he is not concerned entirely with manual labor; consequently a full training in the performance of such sequences must be regarded as forming an essential part of any education worthy of the name. Moreover, the full appreciation of such processes has a higher value than is contained in the mental training involved, great though this be, for it induces an appreciation of intellectual unity and beauty which plays for the mind that part which the appreciation of shape and color plays for the artistic faculties; or, again that part which the appreciation of a body of religious doctrine plays for the ethical aspirations. . . . Geometry is highly desirable in that the objective bases are so simple and precise that they can be grasped at an early age, that the amount of training for the imagination is very large, that the deductive processes are not beyond the scope of ordinary boys [and girls, I hasten to add—A. H.], and finally that it affords a better basis for exercise in the art of simple and exact expression than any other possible subject of a school course."

II. THE PROBLEM OF INSTRUCTION

Text Teacher Pupil

The purpose I have in mind in the present monograph is to present, for the consideration of teachers, a certain number of definite proposals and methods to be employed in the teaching of geometry. The underlying reason for these proposals is the consciousness that the age in which we live is the age of economy. A word constantly upon the lips of the modern man and woman, whether as teacher, as business man or woman, or as citizen, is the word efficiency. I take it that the essence of efficiency is economy—the closest adaptation of means employed to ends sought. I would not have the reader infer that economy in this sense connotes the saving of either space or time in demonstration. For it is often the case that ideal ends sought are best attained by discovering, i. e. disclosing, the kernel of truth hidden in a mass of chaff. This can frequently be accomplished only by going to the very end of one's thoughts, without evasion or slur; and this completeness in treatment often requires ample room, both in time and space. The unifying principle lies at the heart of the subject; and it is necessary to brush aside the non-essentials and to pierce clear through to the centre. Efficiency connotes another element, correlative with economy; and this is the means employed in achieving economy. The longest way round may well be the shortest way through—even if this be not the formulation of a strict mathematical law! If brevity gives quick results with slow or imperfect understanding, it clearly defeats its own purpose. If completeness and naturalness in procedure, though necessitating more of both time and space, stimulates logic and promotes understanding, thereby it demonstrates the efficiency of those methods as superior to mere temporal and spatial economy. Method, then, is the touchstone of efficiency, no less than economy.

In the second place, it cannot be too strongly stressed that geometry is, properly considered, a branch of logic. The significance of the modern developments in the study of geometry, the contributions of Pasch, Peano, Pieri, Hilbert, and other investigators to the study of the foundations of geometry, is the powerful pressure towards the reduction of geometry to the

plane of abstract logic. Enormous gains have been made through the contributions of the abstract mathematicians; yet it is desirable that we should not lose touch with the tangible realities of the world around us. It is essential that we always hold in mind the "evolutionary character of all life processes, whether in the individual or in the race;" for as Moore says: "All science, logic and mathematics included, is a function of the epoch—all science, in its ideals as well as its achievements."

Geometry is pragmatic. We draw a triangle, discover certain relations between the points, sides, and angles of that triangle. We apply the results thus acquired to all other triangles and verify the accuracy and justness—i. e. the truth, of our conclusions. We find that these conclusions will "work"—hold good—with reference to all triangles. Geometry is the science of the study of geometrical figures, the purpose of which is to arrive at general results from reasonable hypotheses by accurate reasoning.

A first suggestion to the teacher of geometry, then, is this: Familiarize yourself, before attempting to teach the subject of Euclidian geometry, with the leading principles and underlying ideas of the modern schools of geometry. A wide background and sound content of knowledge is really indispensable, nowadays, to a proper understanding of the rôle which Euclidian geometry plays in the new world of many geometries set up by modern investigators. One should read at least one work on the subject of the so-called non-Euclidian geometry—for example, the work of Henry Parker Manning published by Ginn and Company. Difficult though it be, in parts, to read, the famous work of David Hilbert on "The Foundations of Geometry" (Open Court Publishing Company) will open before the teacher of geometry new vista into the larger world of modern geometrical thinking. Lastly, the teacher of Euclidian geometry is wofully backward at the present time in not making rational employment of many of the devices employed in the elegant demonstrations of projective geometry. The reading of such a brief and elementary work on projective geometry as that of William P. Milne (The Macmillan Company), designed for use in schools and colleges, should stimulate the interested and open-minded teacher to the ultilization of methods, drawn from the realm of

projective geometry, in the class-room in the teaching of Euclidian geometry. Clearly, only the methods—in especial, methods of construction—should be employed which are susceptible of metrical proofs, reasonably simple and comprehensible. If the teacher is to have any rule of action, any guiding pedagogical principle, it should be this: Impress upon the pupil the necessity of finding the clue to the mystery—discovering the crux of the problem—recognizing the keynote of the piece.

In my own experience in teaching, I have become accustomed to certain deficiencies in the mental equipment of the pupil developed in any serious encounter with the subject of geometry. When these deficiencies betray themselves in pupils who have studied geometry in the high school, they are often directly traceable, not to the mental weakness of the pupil, but to the pedagogical inadequacy of the teacher. When these deficiencies first exhibit themselves in pupils who have never studied geometry before, I have found that they may readily be remedied through the consistent and continued employment of the correct and natural method of instruction.

It may be instructive to set down these difficulties, so that we may be clearly informed as to just what they are and how they are best to be dealt with. The most frequent fault of the pupil expresses itself in the statement: "I don't know how to begin." The reasons for this are two. In the first place, the pupil has been accustomed to memorize the demonstration; and inability to proceed indicates, first, forgetfulness—mnemonic loss. In the second place, the pupil has not been taught the principles of geometrical analysis and the orderly arrangement of his knowledge of geometrical theorems. If the pupil has been trained to deal with each problem as it arises, in the light of the knowledge already acquired from previous theorems, and to analyze the problem in a scientific manner, many of his difficulties will disappear. Under such a system of instruction it will not be necessary for him to memorize the demonstration in order to reproduce it: he will be able, in general, to reproduce it logically as a consequence of preceding theorems. The pedagogical principle to be invoked is: From the very beginning of his study of geometry, the pupil must be trained to rely, not upon his faculty of memory in reproducing a demonstration, but upon his reasoning faculty

to arrive at it logically from the premises and the knowledge of previous theorems which he mentally has in stock.

Precisely the same answer applies in the case of the other stock excuses advanced by halting pupils. One of them frequently encountered is: "I know the proposition perfectly; but I just can't do it now." Another case is that of the pupil who begins bravely; gets part way through his demonstration; comes to a dead halt; and announces that he can proceed no further, as he doesn't know what is the next step in the demonstration. This means, as a rule, that he has been unsuccessful in the attempt to memorize the entire demonstration; he has memorized part of it, but the remainder escapes him—because he memorized the steps in the demonstration, and not the reasons for the steps taken.

This fault—I cannot stress it too strongly—is to be laid, in very large measure, not so much at the door of the school teacher as at the door of the writer of text-books. The methods employed in the texts, for the most part, are methods characterized by compression, elegance, rigor. But the point to be observed is that they are not characterized by naturalness, by accessibility to quick mental reaction, by that sort of evolutional inevitableness so pre-eminently desirable in all scientific procedure and record. I shall concern myself then, in this monograph, with methods designed to remedy the pedagogical faults of the great majority of texts, however impeccable they may be from the standpoint of pure logic. It is no particular credit to a geometer to say that his text-book is logical. An illogical geometry would be an anomaly—a work self-condemned. The prime virtue of a geometry, from the pedagogical point of view, is not that it be logical—that goes without saying; is not that it be understandable—that, too, goes without saying, presuming a reasonable amount of study; but that it shall be *inherently consecutive*. Here the texts, even the best of them, persistently err. A proposition, innocent looking enough, is placed before us. The steps proceed in natural order, one following the other as one would expect; and then suddenly: halt! No thoroughfare. We can, apparently, proceed no further on the basis of the data at our command. The author arbitrarily and mysteriously directs, without hint or explanation, rhyme or reason; From line A B

lay off an angle A B C equal to the angle D E F, and let B C intersect G H at K. From this point on, each step is a logical consequence of the preceding steps; and before we are aware of it, the author suddenly announces:

$$\therefore \triangle ABC = \triangle DEF$$

Q. E. D.

The pupil is as much startled and mystified as the child confronted with the rabbit which the magician draws from his obviously empty hat. Such cases, appearing at frequent intervals throughout any text in geometry, largely explain the dilemma of the pupil who sets down part of the demonstration and then is arrested—because he has not understood the reason for the step which the author has arbitrarily directed him to take. In a subsequent section I shall take up this question in detail, and exhibit the method by which such defects of the texts may be obviated. But I would not be understood, by any means, as wholly absolving the teacher of blame. It is the function of the good teacher, proceeding upon approved pedagogical principles, to analyze and justify any and every step in a demonstration, especially those which have at the moment no obvious justification. The failure to follow this course, on the part of the teacher, is as blameworthy, as is the omission, on the part of the author, to give any suggestion as to the reasons controlling the direction given the pupil.

For the high-school teacher, in especial, this caution is pertinent: Avoid the pitfalls of over-explaining the subject. It is explanation of technic the pupil requires—not explanation without limit or cessation of every problem or difficulty that arises. Once the technic is grasped by the student, he should be encouraged to discover his own reasons for certain steps; and towards the close of his study, he may well be required to supply the argument for every step of the proof, the references to preceding articles being omitted from the text.

The geometer works with tools in the same sense in which a carpenter works with tools. There are a certain number of constructions, incessantly employed, which he must be able to effect with the same accuracy and efficacy exercised by the carpenter in the use of hammer, saw, adze, plane, and drill. In the course of long experience in the teaching of geometry, I have

been vividly impressed with the almost invincible weakness of the pupil in broad and comprehensive knowledge of the scope and use of the tools at his command. He is taught one way only, as a rule, to perform these fundamental operations. And this, too, in face of the paramount consideration that these very fundamental operations, by reason of their incessant employment, should be better understood and more intensively studied than many of the individual theorems, the demonstration of which require their use. In a subsequent section, I shall consider in detail the principal fundamental operations; and there give a group of alternative constructions, thus revealing the richness and variety of geometric structure and form, as instrumentalities of demonstration and construction. By resorting to the elegant methods of projective geometry in certain of these constructions, and deriving therefrom certain metrical principles in construction, the content of Euclidian geometry is greatly enriched and amplified.

III. ANALYSIS VERSUS SYNTHESIS

In approaching the broad field of geometrical instruction, the subject naturally falls into two divisions: theorems and problems. The theorems are certain fundamental and indispensable propositions of relation, which are fully set forth in logical and orderly demonstration. They are phrased as dogmatic assertions; and the demonstrations set forth are proofs of these assertions. The problems are similar to the theorems, in that they represent truths which may be arrived at; they differ, in the main, from the theorems in being less rich in content and less frequent of application. Many such problems or exercises, so-called, are really problems in mensuration—numerical, arithmetical in nature. They are elementary examples in applied mathematics; except for the practice in the use of geometrical machinery, their solution does not markedly increase the power of the student or enrich the content of geometry, since, once solved, they lose their interest because of lack of generality. To solve one such enables one to solve not new problems, but only one just like this one. Exercises of this character have their use, however; since in the first stages of the study of geometry, it is desirable for the student to have frequent practice in the simplest type of exercise, namely the numerical type.

The real field of geometrical study lies in the infinite number and variety of exercises, general in their nature, fertile in their content and suggestiveness, which nature in all her various manifestations affords for the employment of the mind of man. There are two general methods of approaching geometry. One makes paramount the study of the propositions; and relegates the exercises to a very minor and subsidiary position. The other gives abundant attention to the study of the propositions; but regards such study, not as an end in itself, but essentially as preliminary to the study and solution of problems. It must be clear, from the principles of teaching which I have enunciated above, that the second method is, in my judgment, the true pedagogical principle. The genius of the American nature is inventive, constructive; and under the American method the teacher of mathematics naturally considers geometrical exercises, not as "riders," after the English fashion, but as the grist of the mill of the empirical

geometer. The American teacher of geometry, if he or she run true to form, regards the fundamental propositions as tools, instruments, for building new and intricate structures, and for revealing hidden relationships in interconnected figures. Consequently, it is eminently desirable that the teacher should examine the fundamental constructions and basic propositions from many different points of vantage, to enable the pupil not only to understand the means to be employed in their demonstration, but—and this is all important—to understand fully the essential purport, significance, and utility of the geometric truths unfolded.

The method commonly employed in text-books on geometry is the method of synthesis. The proof is put together as we fit together a set of interlocking blocks. One block is set up; next to that is placed that one block which precisely fits the outline of the first; and so we proceed until the structure is complete. The method is logical; it is elegant; it is concise. The reasoning is from the general to the particular truth; and the specific fact is inferred from a general law, through a chain of logical consequence. In the case of a large number of theorems, the method has demonstrated itself to be the best method—having been employed from the time of Euclid to the present. In the case of many other theorems, the synthetic method is often ineffective and confusing; Schopenhauer in his *Welt als Wille und Vorstellung* objects to such "jugglers' pranks," as he called them, in which "the truth usually comes in through the back door." The synthetic is not the method of research: it is the method of tabulation; it is not the method of discovery: it is the method of recording the steps in the most succinct manner after the truth has been discovered. Synthesis arrives logically and expeditiously at a result; yet it often leaves us in the dark regarding the process, or at least certain steps in the process, by which the result is arrived at. Synthesis gives us the truth; but it often fails to put in evidence the reasoning by which the truth was originally discovered.

Analysis is the method of discovery, of research. It is the opposite of synthesis—being essentially the process of taking a thing to pieces instead of putting pieces of a thing together. The procedure in analysis is the inverse of the procedure in synthesis.

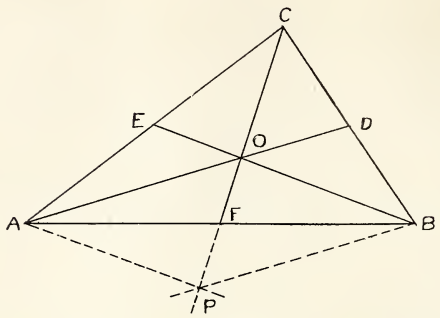
In analysis, we reason from parts to the whole, from the particular to the general. We assume the conclusion we suspect or divine to be true; and reason backward to established truths. Since the underlying truth is the same, whether the analytic or synthetic method be employed, it is clear, that once a valid consequence is reached through analysis, this consequence may then be employed as the basis of a synthetic proof.

A mode of analysis employed in the Euclidian geometry is the method known as *reductio ad absurdum*. If insuperable difficulties arise in the analysis, based upon a given assumption, then we assume its contradictory; and by analysis, if this be shown to be false, then the assumption first posited is thereby demonstrated to be valid. Direct proofs are generally preferred, although logically the method of *reduction of absurdum* is flawless.

In the teaching of geometry, even those theorems which are best treated synthetically should be subjected to careful analysis; for in this way the student is enabled to gain a deeper insight into the mechanism of demonstration. In the case of theorems in which synthesis is clearly artificial and certain steps have the character of the arbitrary or the adventitious, such proof should be rejected outright; and an analytic demonstration supplied. This is the true, the natural, way to develop the power of investigation on the part of the pupil. By such analytical proofs the student is enabled to acquire the technic of demonstration. De Morgan exemplifies the wisdom of this pedagogical principle in the pregnant observation: "It is as necessary to learn to reason before we can expect to be able to reason, as it is to learn to swim or fence, in order to obtain either of those arts."

In order to illustrate the method of analysis, the following theorem is chosen because of its interest.

The medians of a triangle are concurrent in a trisection point of each.



Let A B C be the given triangle and A D and B E the medians drawn from A and B to the middle points D and E of the opposite sides B C and A C respectively.

Since A D lies between A B and A C, and B E lies between B A and B C, A D and B E must intersect at some point O within the triangle.

Let us join C to O and attempt to prove that C O extended to meet A B, will intersect it in its mid-point F. For in that case C O F, drawn to pass through O, the meet of A D and B E, and the mid-point F of A B, will be the third median of the triangle.

If F were the mid-point of A B, then a parallelogram might be formed by extending O F to P, making F P and O F equal; in which case A P would be parallel to E O and B P parallel to A O.

Accordingly, let us draw a line through A parallel to E O meeting C O extended at P. Then since C E = E A, therefore C O = OP, since a line (E O) passing through the middle point (E) of one side (A C) of a triangle (A P C) parallel to the base (A P) passes through the middle point (O) of the other side (C P).

Join P to B with a straight line P B. Since C O = O P and C D = D B, therefore B P is parallel to O D, since a line (O D) joining the middle points (O, D) of two sides (C P, C B) of a triangle (P C B) is parallel to the base (B P).

Hence A O B P is a parallelogram, since we have proved that the opposite sides in pairs (O B, A P; O A, B P) are parallel. And consequently F is the middle point of A B, since the di-

agonals (O P, A B) of a parallelogram (A O B P) bisect each other.

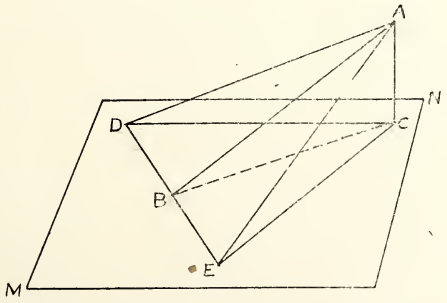
Moreover A P = 2 E O, since the line (E O) joining the middle points (E, O) of the sides (A C, P C) of a triangle (A C P) is equal to one-half the base (A P). Likewise B P = 2 O D. But A P = O B, and B P = O A, since the opposite sides of a parallelogram are equal. Hence O B = 2 O E, and O A = 2 O D. Also O F = ½ O P = ½ O C, since O P = O C.

Since the consequences following from the assumptions are true, a synthetic proof may readily be established, first by drawing a line through A parallel to E O meeting C O at P, and then joining P to B.

The superiority of such analysis as that outlined above, over the synthetic proof found in the text-books (See Beman and Smith's *Plane and Solid Geometry,* page 75), is in making clear the grounds for the constructions made, *at the time they are made.* Instead of arbitrarily executing the constructions and afterwards explaining the reasons, the reasons are given in advance of the construction itself.

Another illustration, chosen from solid geometry, will elucidate the matter still further. Take the following theorem:

If a line intersects a plane, the line in the plane perpendicular to the projection of the first line at the point of intersection, is perpendicular to the line itself.



Given the line (A B) interesecting the plane (M N) at B, C the projection of A on M N, and D E perpendicular to B C at B.

We wish to show that D E is perpendicular to A B. If B A were perpendicular to a line at its middle point, then every

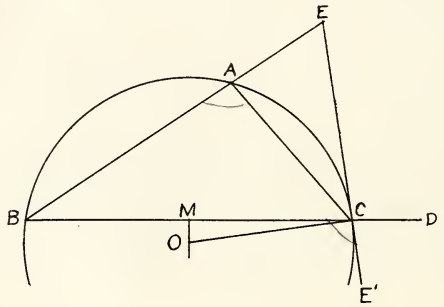
point in B A would be equally distant from the extremities of the line. If D and E be chosen equally distant from B, then A D = A E; and in consequence D C = D E, which is true if B is the mid-point of the line D E, since B C is perpendicular to D E.

Retracing our steps, a synthetic proof may be at once established, by the following steps: (1) Lay off B E = D E; (2) D C = E C; (3) A D = A E; (4) hence A B is perpendicular to D E at B.

A last illustration of the method of analysis, which I have not observed elsewhere in print, brings out very clearly the technic of the analytic method.

The theorem to be demonstrated is the extremely useful theorem:

The exterior angle of a triangle is equal to the sum of the two opposite interior angles.



Given the triangle A B C; to show that angle A C D is equal to the sum of interior angles A and B.

In order to prove this, a line may be drawn through C laying off, either from C D or C A, an angle equal to A, and then showing that the remaining angle is equal to B.

Rejecting the ordinary treatment, as found in the text-books, let us draw C E, making the angle D C E equal to angle B A C. It then remains to be shown that angle A C E equals angle A B C.

Now since angle B C E′ equals angle B A C, the circle circumscribing triangle A B C will touch C E′ at C. For a tangent (say C E′) to the circle circumscribing the triangle A B C, at the point C will make the angle B C E′ equal to the angle B A C, since each is measured by one-half the arc C B (*not* arc C A B).

Construct the center O of this circle at intersection of perpendiculars to E′ C at C and to B C at its middle point M. Consequently the angle A C E equals the angle C B A, each being measured by half the arc A C.

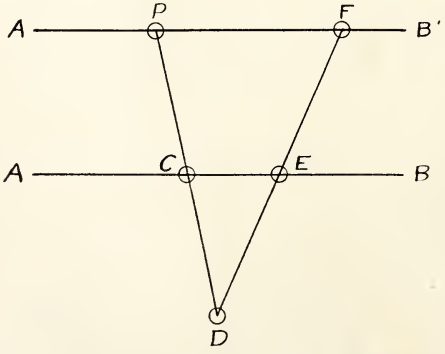
In conclusion, I would observe as the result of my own experience in teaching geometry, that this method of analyzing the theorems, taking up the reasons for the different steps, and studying the various avenues of approach to a solution, always proves genuinely stimulating to the pupil. He soon becomes animated with the desire to tackle theorems on his own account; and not infrequently develops at an early period in his study marked power and acuteness in analysis.

IV. THE BASIC PROBLEMS OF CONSTRUCTION

In an earlier section, I have called attention to the desirability of understanding thoroughly the use of the tools we are in the habit of handling. No carpenter would be entrusted with the building of a house if he did not know how to drive a nail or plane a board. In the same way, the geometer—and by geometer I mean the pupil who is working in geometry—should be thoroughly familiar with the geometrical constructions he is most frequently called upon to employ.

In the remainder of this section, I shall give methods for performing these fundamental operations of construction—operations not ordinarily encountered in texts in Euclidian geometry. The claim is not made for these constructions either that they are simpler than the constructions in general use, or that they should displace these conventional modes. Indeed, certain of these constructions are somewhat complex in theory; and they are interesting in themselves as problems. It should be clear, however, that these, and other constructions which might be given, should actually be effected by the pupils in certain problems definitely assigned by the teacher—in order to bring out the essential features of the given construction and also to widen the geometrical horizon of the pupil.

1. *Through a given point to draw a line parallel to a given line.*

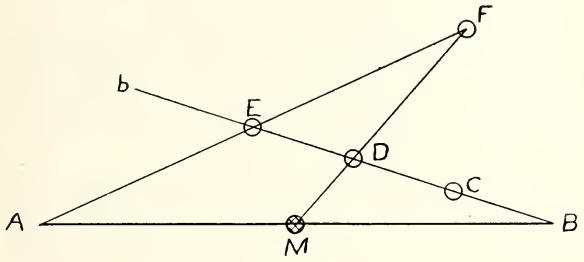


Through P, the given point, draw a line in such a direction as to cut the given line, A B, in C; and lay off on it C D = P C. Through D draw a line other than D P, meeting A B at E; and

lay off on it E F = D E. Then P F is the required line—since the base of a triangle is parallel to the line joining the middle points of the sides.

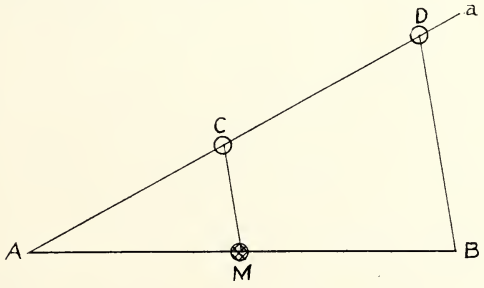
2. *To bisect a given segment.*

First Method.



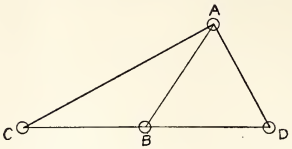
Through B, an extremity of the given segment A B, draw any line *b* not coinciding with A B, and lay off on it from B any three equal lengths B C, C D, and D E. Draw A E and prolong it to F, making E F = A E. Then F D prolonged will bisect A B at M. The reason is obvious, if we join B to F, and recall the theorem that the medians of a triangle (A B F) intersect in a point (D), two-thirds of the way from each vertex to the opposite side.

Second Method.



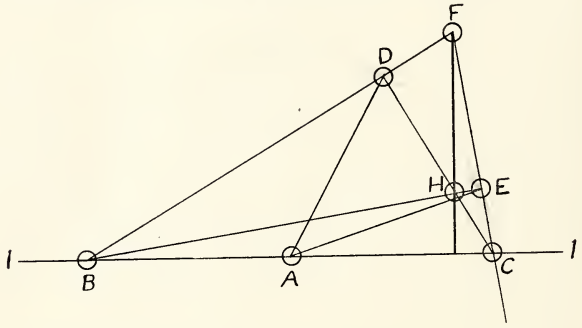
On any line through A, extremity of segment A B, not coinciding with A B, lay off any two equal lengths A C and C D. Join D to B; and by Construction 1 draw through C a line parallel to D B, which will intersect A B at its middle point M. This is true, since a parallel to the base of a triangle through the middle point of one side will bisect the other side.

3. *To construct a right angle at a given point.*



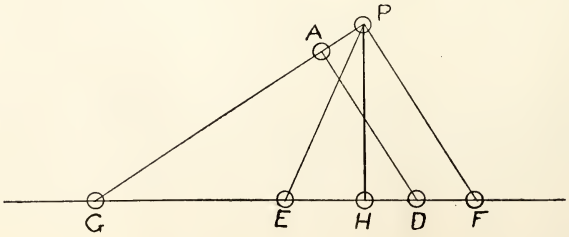
Through A, the given point, lay off a given segment A B in any direction; on any other line through B lay off B C = B D = B A. Then angle C A D is a right angle. The proof is left to the reader.

4. *To draw a line perpendicular to another line.*



From any point A on the given line 1 1, lay off two equal segments A B, A C on the line; and two other equal segments on two lines through A such that A B = A D = A E = A C. The line joining F, the intersection of B D and C E with H, the intersection of B E and C D, will be perpendicular to the line 1 1. This follows since D and E lie on the semi-circle on B C as diameter; and since C D and B E are perpendicular to the sides B F and C F, respectively, of the triangle B C F, F H is the perpendicular to the side B C.

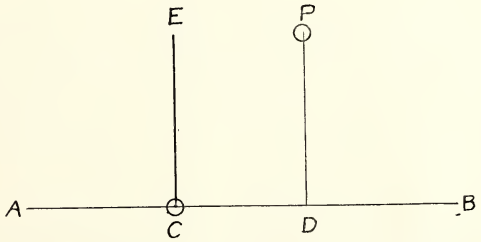
5. *To draw a perpendicular from a point to a line.*



Construct a right angle G P F at the given point, P (Construction 3, above), the sides of the right angle meeting the given line at G and F. Lay off G D = G P. Through D draw D A parallel to F P (Construction 1, above). Lay off G H = G A. Then P H is perpendicular to the given line.

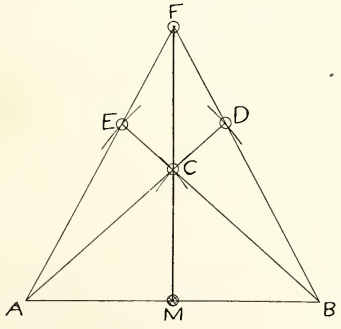
The demonstration of the validity of this construction furnishes an excellent illustration of analysis, and is suggested as a diversion for the reader.

6. *To erect a perpendicular to a line at a given point in the line.*



From P, any point outside the given line A B, drop a perpendicular P D to the line (Construction 5, above). At C, the given point in the line, draw a parallel C E to P D (Construction 1, above). Then C E is perpendicular to A B at C.

7. *To bisect a given segment at right angles.*
First Method.

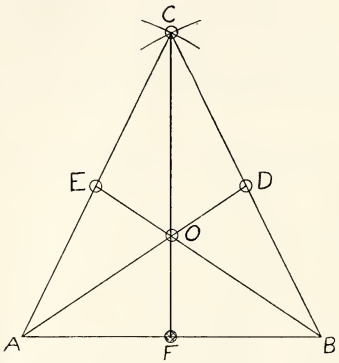


Describe equal arcs from A and B, extremities of given segment, intersecting in C, and draw straight lines A C and B C. Describe other equal arcs from A and B, intersecting A C and B C at

D and E. Draw B D and A E intersecting at F. Then F C is perpendicular to A B at M, its middle point.

The proof follows from the equality of the triangles A B D and A B E, which is true because A B C is an isoceles triangle; and hence A B F is an isoceles triangle.

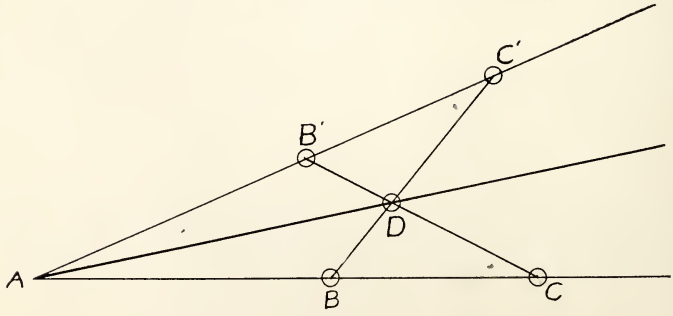
Second Method.



Describe equal arcs from A and B, the extremities of the given segment, intersecting at C; and draw the straight lines A C and B C. Bisect the two segments A C and B C at D and E, respectively (Construction 2, above). Draw A D and B E, meeting at O. Then C O bisects at right angles the segment A B at F.

The proof follows from the theorem concerning the medians of an isoceles triangle. It is to be observed that Construction 7 combines into one, two constructions already given above—viz., to bisect a given segment and to erect a perpendicular to a given line.

8. *To bisect a given angle.*



On the sides of the given angle A, lay off A C′ = A C, A B′ = A B. Draw straight lines B C′ and B′C, intersecting at D. Then the line A D is the required bisector of the angle A.

The proof follows at once from a comparison of three pairs of equal triangles: A B C′, A B′ C; D B C, D B′ C′; A D C, A D C′.

This list of fundamental constructions may readily be extended. Other fundamental constructions will suggest themselves to the teacher of geometry.

It is recommended that the teacher of geometry, into whose hands this monograph falls, require the pupils in his or her classes to execute a considerable number of constructions, employing the methods 1-8 given above. Familiarity with these and other methods and facility in their use may best be acquired through actually employing them when such constructions are called for. A stronger grasp of fundamental geometry will thus be secured.

V. THE PROBLEM OF RESEARCH

The study of the theorems of geometry, the analysis of the various steps in the demonstrations, the investigation of these classic theorems from various points of view—all constitute valuable training in the acquisition of geometrical knowledge. But the theorems should never be studied for themselves alone, relegating the study of the other branches of the subject to a subsidiary position. The object of the study of the theorems is not simply to acquire a given body of knowledge, desirable as such an attainment may be. The object of such study is the mastery of the more fundamental principles of the subject, in order to acquire the indispensable technic for research. The deeper purpose of study is the acquisition of power in the employment of a specific technic for the mastery of new, unfamiliar, and original forms and types.

In all forms of investigation, analysis is the most fruitful instrument for the discovery of truth. This is particularly the case with a subject like geometry, in which demonstration resolves itself into a logical sequence of accepted syllogisms. It is necessary, at the outset, to resolve the problem into its simpler elements. Once the useful relationships have been selected, viz., those which are both necessary and sufficient for the task in hand, the desired truth may be arrived at. It is often the case that the pupil is able to segregate certain statements of relation; but if these, while being necessary to the solution, are not sufficient, either in number or content, then the pupil is unable to arrive at significant results. It is obvious, then, that a thorough knowledge, and comprehensive insight into the nature, of the fundamental theorems is absolutely indispensable, if the pupil expects to succeed in the often difficult and complex task of the solution of original problems, which is in reality a veritable, if elementary, branch of individual research.

In all scientific study and investigation, method and technic are almost as necessary as knowledge; for to attain a mastery in method and technic means, essentially, to so systematize one's knowledge as to make it effective in the often ardous task of arriving at truth. Consequently, the second phase of the study of geometry, after mastering the fundamental propositions, is to

enumerate and order, in a systematic way, the various methods, characterized by a measure of generality, which may be employed with success in investigation. No golden rule or unique formula has been or can ever be devised for the solution of all geometrical problems. Even were it conceivable that such rule or formula could be devised, it would be a source of regret rather than of satisfaction—since thereby the study of geometry would be shorn of one of its supreme merits, the training of the higher reasoning faculties, and would degenerate into the merely formal application of the general rule or formula. The discovery of co-ordinate representation approximates nearest to a universal form of solution; but its employment without exception in all geometrical problems would rob Euclidian geometry of its true, inherent character—as geometry, instead of a branch of geometrical algebra. At the same time, it is clear that methods virtually identical with those of co-ordinate representation are found to be indispensable in the treatment of certain geometrical theorems; and in many others, the algebraic or co-ordinate method possesses an elegance and simplicity not always attainable through pure geometrical technic.

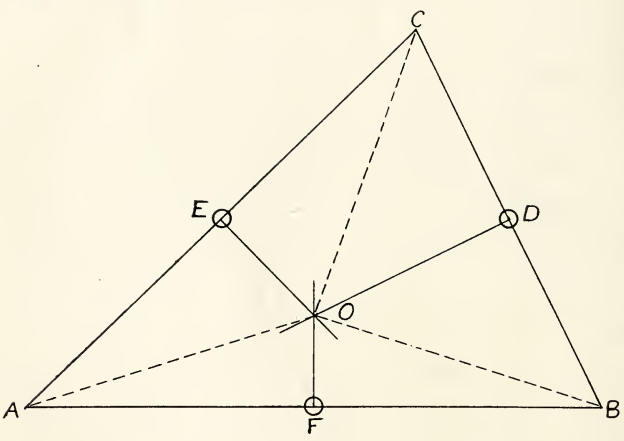
In this section, it shall be my purpose to catalogue and explain a certain number of methods or modes of approach to the solution of geometrical problems. A mastery of these various methods and the acquisition of facility in their application will prove of almost immeasurable value to the student of geometry. No claim, of course, is made that these are the only methods which may be employed; or even that they are, in all cases without exception, the best methods to be employed. But it is certain that a thorough grasp of the principle, mechanism, and technic of these methods will heighten, in an exceptional degree, the individual's power of investigation; and impart to the student, versed in their use, a mental confidence and an intellectual power denied to the ordinary pupil of geometry who is not versed in these methods. Experience in the teaching of geometry has convinced me that the text-books in general use are, almost without exception, wofully deficient in this very respect—in that no systematic examination of method and technic is made in these texts. The lack of such machinery is, it can scarcely be doubted, in large measure responsible for the comparative failure to de-

velop a higher and more effective type of investigator, of an elementary type, out of our classes in geometry, in the high school, the college, and even in the university. At the risk of criticism for taking a position which may appear to be radical, I assert that no teacher, however enthusiastic or capable, should undertake to require of his or her pupils in geometry the solution of a great number and variety of geometrical problems and exercises, not merely of the numerical or mensurational type, until such teacher has given to the class a full and scientific exposition of the fundamental methods and basic technic of geometric investigation.

1. *The Method of Analysis.*

The principles of this method have already been explained. A simple example in this place will re-inforce the general ideas involved in this method. To prove the following:

The perpendiculars erected at the middle points of the sides of a triangle intersect at a common point.



Analysis.

Erect perpendiculars at the middle points, F and D, of the sides A B and B C of the triangle A B C, intersecting at O; and from O drop a perpendicular upon the side A C.

Now if E is the middle point of A C, then O A would equal O C. But O B = O A, and O B = O C; hence O A *is* equal to O C.

Now reverse this process and we have the following:

Synthesis.

O A = O B, O C = O B (*s. s. a = s. s. a*). Hence O A = O C; and O E, the altitude of an isoceles triangle, bisects its base A C in E, the middle point.

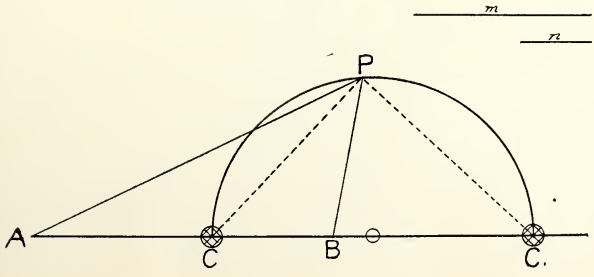
In this connection, it may be remarked that a similar proof might be effected by drawing O E to bisect the angle A O C.

2. *The Method of Successive Substitutions.*

The method of analysis is first found in geometrical literature in Euclid's Geometry; but it has generally been attributed to Plato. A form of analysis, modern in origin, is known as the method of successive substitutions. It has its familiar analogue in the elaborate "hypothetical question" of the contemporary law courts. In place of the theorem to be proved, the student substitutes another upon which that depends; and continues this process until he arrives at some theorem known to be true in the light of the given hypothesis. Thus the student reasons: "Statemen A is true if statement B be true; and statement B is true if statement C is true. But statement C *is* true; and hence statements B and A are true." Another way to put it, similar to the first, would be: "I can prove Theorem A if I can prove Theorem B; and I can prove Theorem B if I can prove Theorem C. But I *can* prove Theorem C; and therefore Theorems B and A logically follow."

This method may happily be illustrated by the following interesting problem:

The geometric locus of all points, the distances of which from two given points are in a fixed ratio m : n (say m > n), is the circumference of a circle having its centre on the line joining the two points, and dividing that segment in extreme and mean ratio.



Given A and B the two fixed points and P any point on the locus, such that $P A : P B = m : n$.

P lies on the circumference, having $C C_1$ as diameter, when C and C_1 divide A B in extreme and mean ratio, IF the following statements be true:

$$A C : C B = A C_1 : B C_1 \quad \ldots\ldots (1)$$

$$\text{Angle } C P C_1 = 90° \quad \ldots\ldots (2)$$

But equation (1) IS true, since if C and C_1 ARE points on the locus

$$A C : C B = m : n$$

and

$$A C_1 : B C_1 = m : n$$

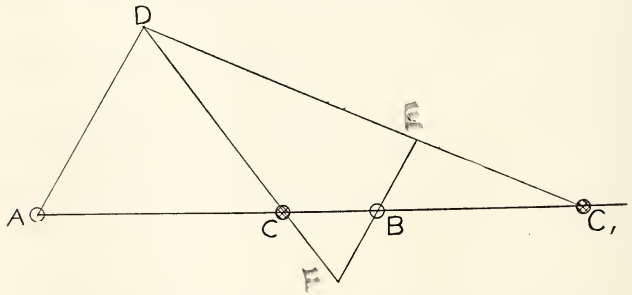
and therefore

$$A C : C B = A C_1 : B C_1.$$

And equation (2) IS true, since if we join C and C_1 to P, C P and C_1 P are the internal and external bisectors of the angle A P B, and these are at right angles to each other.

The points C and C_1 are said to divide the segment A B harmonically in the ratio $m : n$. In consequence, the problem resolves itself into a solution of the following problem in construction:

To divide harmonically a given segment in a given ratio.



To do this, lay off the length A B on any given line; and on two parallel lines drawn in any direction through A and B lay off A D and B E equal to (or in the ratio of) m and n, respectively. On the line E B extended lay off B F equal to B E. Draw D E and D F meeting A B internally and externally in C_1 and C, which are the required points. This follows at once

from the similarity of the pairs of triangles A D C, B F C; A D C_1, B E C_1.

This is the famous Problem of the Lights, which may be stated in physical terms as follows:

Given two lights of different intensities, to find the locus of the points of equal illumination on a plane which embraces the line of the two luminous points.

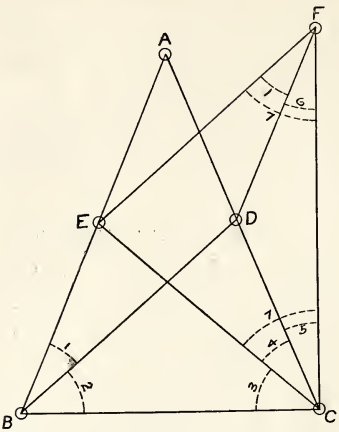
3. *The Method of Reductio Ad Absurdum.*

In certain problems, it is sometimes the case that a direct method of solution is very difficult to discover. In this case, resort may be had to what is known as the indirect method. The pupil begins by cataloguing all the relationships, including the given theorem and its contradictory or contradictories. Assuming each contradictory in turn to be true, a chain of syllogisms will eventually result in the denial of the initial assumptions, and hence each contradictory of the given theorem is false. In consequence the one remaining alternative, viz., the theorem in question, must be accepted as true.

A problem of extraordinary historic interest, which is found guilelessly proposed by Wentworth among easy exercises (*Geometry,* Exercise 43, p. 42), is the following:

If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

The difficulty of effecting a direct demonstration is illustrated by an alleged proof seriously advanced by Loomis in his *Original Investigation* (pp. 10-12), which is grossly fallacious in its reasoning. Many other direct "proofs" which are in print are equally fallacious. The following indirect proof may prove of interest to students and teachers:



Given B D and C E, the bisectors of the base angles A B C and A C B of the triangle A B C, to be equal.

To prove angle A B C = angle A C B.

Now there are three possible suppositions:

Angle A B C greater than angle A C B.....(1)
Angle A B C less than angle A C B.....(2)
Angle A B C equal to angle A C B.....(3)

Let us assume angle A B C greater than angle A C B, i. e., angle 2 greater than angle 3. Also we have

Angle 1 = angle 2; angle 3 = angle 4.

Now since angle 2 is greater than angle 3, C D is greater than B E. (If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second). Now through D and E draw lines parallel to B A and B D respectively, intersecting in F, and join F to C. Then B D F E is a parallelogram, having two pairs of opposite sides parallel. Consequently E F = B D = E C; angle E F D = angle 1; angle E F C = angle E C F. But angle 1 is greater than angle 4; and hence angle 6 is less than angle 5. Hence C D is less than D F. [If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle.]

Therefore C D is less than B E, since D F = B E. But this is impossible, since as we have shown above C D is greater than B E. Accordingly angle 2 is not greater than angle 3, i. e., angle A B C is not greater than angle A C B.

In like manner, by drawing a similar auxiliary figure to the left of the triangle A B C, and assuming angle A B C less than angle A C B, i. e., angle 3 greater than angle 2, we find ourselves involved in a similar contradiction. Hence it follows that angle A B C is not less than angle A C B.

Since of the only three possible suppositions, we have shown that (1) and (2) are false, hence it follows that (3) must be true, that is:

Angle A B C = angle A C B.

Consequently A B = A C, and the triangle A B C is isosceles.

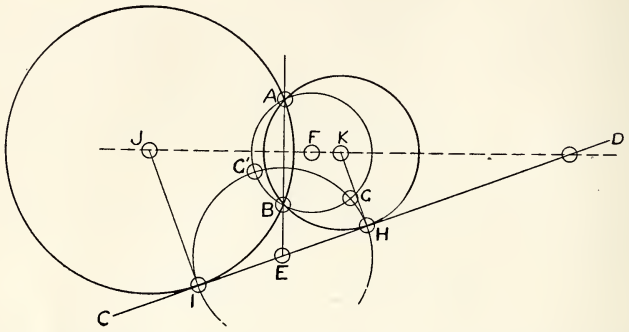
This problem has an interesting history; and it has been traced back to the *Nouvelles Annales* for the year 1842. Discussions of it may be found in the *London, Edinburgh, and Dublin Philosophical Magazine* for the years 1852, 1853, 1874; in the *Lady's and Gentleman's Diary* for the years 1856, 1857, 1860; in the *Intermédiaire des Mathématiciens* for 1895; and in the *American Mathematical Monthly* for the year 1900. In a subsequent publication, I purpose giving a general survey of this now classic problem, with various methods for its solution.

4. *Method of Intersection of Loci.*

In this case, it is necessary to determine the positions of the sets of points determined by the separate conditions of the problem. Each condition will lead to the construction of at least one locus; and the intersections of these various loci, corresponding to the simultaneous satisfaction of the given conditions, will constitute the desired points. Thus, to find the centre of a circle passing through three given points, A, B, C, we determine: the locus of all points equidistant from A and B, and the locus of all points equidistant from B and C (or A and C). The intersection of these two loci constitutes the desired point.

An interesting problem in construction is the following:

To describe a circle through two given points and tangent to a given straight line.



Given two points, A and B, and the line C D. To describe a circle passing through A and B, and touching C D. If now we join A and B with a straight line, prolonged to meet C D at E, we know that E A . E B = square of the tangent from E to the point of contact with the required circle. Moreover, we know that E A . E B = square of the tangent from E to *any* circle through A B. Hence, choosing any point F on the perpendicular bisector of A B as centre, describe a circle passing through A and B. Then the tangent E G (or E G') from E to this circle will be equal to the tangent from E to the required circle; and as the point of tangency lies on the line C D, it must lie on the two loci: (1) A circle with E as centre and E G (or E G') as radius; and (2) the straight line C D. Hence H and I are the required points of tangency. It remains to find the centre of the required circle. Since our construction gives us two points of tangency, there are two circles fulfilling the required conditions. The centres of these circles lie on the following loci: (1) The perpendicular bisector of A B; and (2) lines perpendicular to C D at the points H and I. Constructing the latter lines, the intersections stated, viz., K and J are the centres of the required circles.

As an illustration of the pitfalls, which imperfect or inadequate analysis conceals, it is comical to observe that Stone, in his *Method in Geometry,* "scientifically" applying the "method of analysis" to this same problem in construction, arrives at only

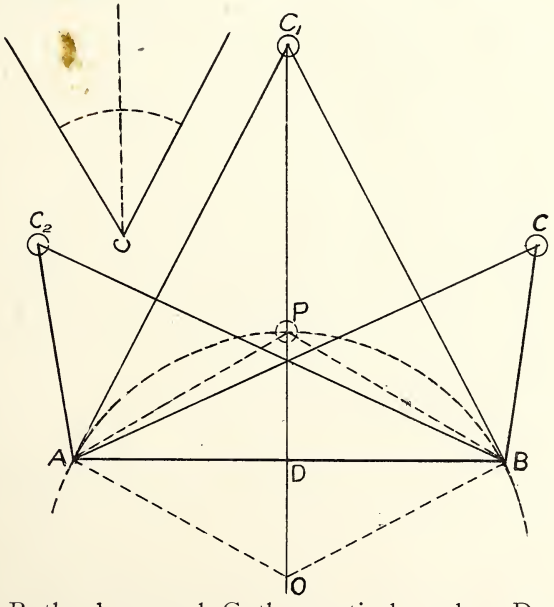
one circle fulfilling the required conditions—although the figure he gives eloquently calls for two! The case having but one solution is trivial.

5. *Method of Construction of Loci by Points.*

It sometimes happens that the required locus cannot be determined at once from the given conditions. But the construction of a few of the points obviously lying on the locus, may suggest that the locus has a definite form, and position; and a little further analysis, thereafter, will usually give the needed proof. This method is probably of French origin; and finds its starting point in the plotting of certain readily discovered points on the locus, known as *remarkable points*—that is, points which may readily be remarked or noted.

As an illustration of this method, the following problem is characteristic:

Given the base and the vertical angle of a triangle to find the locus of its in-centre.



Given A B the base and C the vertical angle. Drawing the bisectors of the base angles, A and B, these intersect in P, the in-centre. When C is at A, P also is at A; when C is at B, P also

is at B; and when the triangle A B C is isosceles, with A C = B C, then both C and P are on the perpendicular bisector of A B. It may be surmised, from the position of these three remarkable points, that the locus is the arc of a circle. Now angle A P B is the supplement of one-half the sum of the base angles A and B, that is the supplement of $90° - \frac{C}{2}$, that is, $90° + \frac{C}{2}$. Since this angle is a constant, because angle C is a constant, it follows that the locus of P is the arc of a circle passing through A and B. The centre O of the circle is found as the intersection of the loci: (1) The perpendicular bisector P D of A B; and (2) a line through A making with A B on the side opposite to P an angle $\frac{C}{2}$. This last follows from the consideration that when the triangle A C B is isosceles, angle C A B equals $90° - \frac{C}{2}$; and since angle O P A $(= 45° + \frac{C}{2})$ equals angle O A P, the angle O A D $= (45° + \frac{C}{4}) - (45° - \frac{C}{4}) = \frac{C}{2}$.

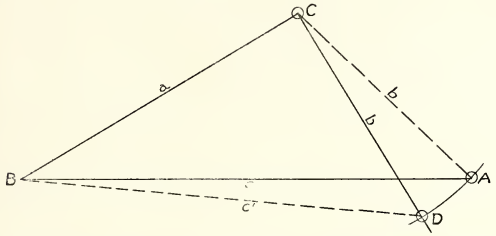
6. *Method of Transformation; Construction of Auxiliary Figures.*

In the case of many problems met with in practice, analysis of the figure as drawn may not at once suggest, or even lead to, a solution of the problem. In that case, it is desirable to analyze the figure in the light of the given relationships; and so be led to the construction of auxiliary figures suggested by the given relationships. Thus, on drawing the new lines or figures, we transform the given problem to another problem upon which it depends. Problems of this character, cryptic upon the surface, will tax in a high degree the igenuity and imaginative powers of the student. This method should be studied with particular care in any course in geometry; but difficult problems requiring the use of this method for their solution should not be given the pupil until he has acquired considerable facility in analyzing and solving geometrical exercises.

A simple illustration of this method will be found in both of two alternative methods of solving the converse of the Pythagorean Theorem:

If the sides of a triangle are connected by the relation $a^2 + b^2 = c^2$, *then the triangle is right-angled at C.*

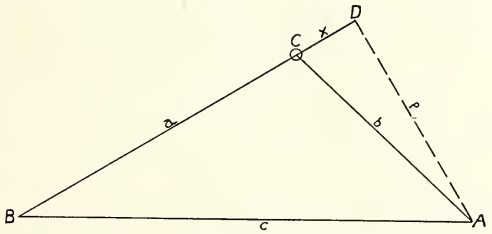
Solution a.



Given the triangle A B C, with the sides connected by the relation $a^2 + b^2 = c^2$.

Now, by the Pythagorean Theorem, drawing C D perpendicular to B C and laying off C D = C A, we have $\overline{BC}^2 + \overline{CD}^2 = \overline{BD}^2$, or $a^2 + b^2 = c'^2$. Hence $c = c'$, and c coincides with c^1 in position, since the position of c′ is fixed.

Solution b.



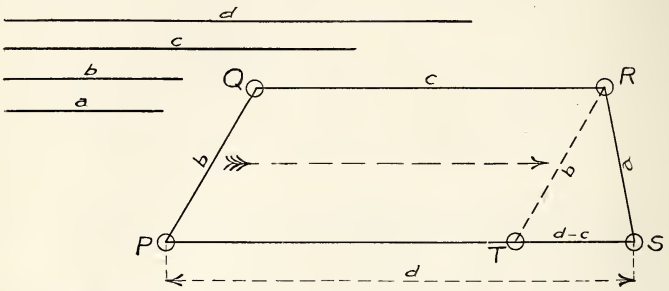
Assuming angle B C A greater than 90°, drop a perpendicular A D from A upon the line B C produced; and designate A D and C D by p and x, respectively. Now $(a + x)^2 + p^2 = c^2$; and $p^2 + x^2 = b^2$. Also $a^2 + b^2 = c^2$. Hence $2ax = 0$. And since a is not equal to zero, therefore $x = 0$; and in consequence, $p = b$. Accordingly D falls at C, and b coincides with p in position, perpendicular to B C at C. Hence it follows that angle C, assumed as greater than 90°, is shown of necessity to be 90°.

7. *Method of Parallel Translation.*

This method may be regarded as a special case of Method 6, above—since it requires the construction of auxiliary figures. In particular, if certain lines or configurations be translated into new positions parallel to the original positions, the new figure

will present the parts in relationships which are known; and analysis will lead to a solution. A simple exercise is the following:

To construct a trapezoid, given the four sides.



Given the four sides *a, b, c,* and *d,* of which *c* and *d* are the parallel sides. Tentatively construct the figure. Then it may be noted that if *b* be translated parallel to itself into the position of the dotted line, a parallelogram P Q R T will be formed; and the remaining triangle T R S will have the sides *a, b,* and *d-c.* Hence the following is the construction required: on P S constructed equal to *d* lay off P T equal to *c,* and with T and S as centres describe arcs with radii equal to *b* and *a* respectively intersecting in R. Through R lay off R Q equal to *c* and parallel to P. S. Then join Q to P by a straight line. Hence P S R Q is the required trapezoid.

It should be noted that the trapezoid may be constructed in a considerable number of ways—constructions which are left to be made by the reader.

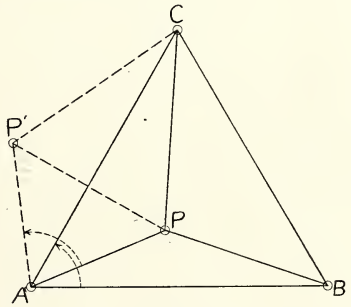
8. *The Method of Rotation. Symmetry.*

However much the abstract logicians and mathematical purists deplore the employment in geometry of the concepts of motion which is a purely physical process, experience demonstrates the utility of such methods. This is the case with pupils first entering geometry; and many fascinating solutions have been effected by the device of folding sheets of paper. In the preceding article, the movement of an object in such a way as always to remain parallel to itself, to which is given the name of translation, was exhibited; and the utility of the method was shown. In a similar

way, figures may be rotated into new positions; and the auxiliary figures thus produced may reveal the hidden properties which are sought. This is found to be desirable in examples in which the parts to be compared are not adjacent, or parts of the same figure; rotation brings the parts into a juxtaposition which enables them readily to be compared. Rotation may be effected about a point, or about a line; in the latter case, the line constitutes an axis of rotation.

Case a. Rotation about a Point.

Given a point P within an isosceles triangle A B C (A B = A C) such that angle A P B is greater than angle A P C, to prove that P C is greater than P B.

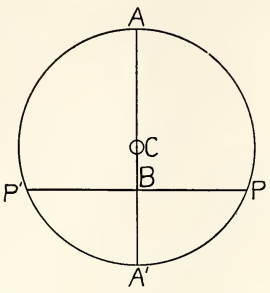


Since the two lines P C and P B must be compared in magnitude, it is clear that P C and P B should be sides of a certain triangle opposite angles, the relative magnitudes of which are known. Let us rotate the triangle A P B about A in the plane in the counter-clockwise direction until it takes up the position A P' C, A B now coinciding with A C. By connecting P and P' with a straight line, we have a triangle P C P' which may afford comparison of the relative magnitude of P C and P' C (= P B). Now angle A P P' equals angle A P' P since A P = A P'. Hence, subtracting these equal angles from the unequal angles A P' C and the lesser angle A P C, we have angle P P' C is greater than angle P' P C. Hence P C is greater than P' C, that is, than its equal P B.

Case b. Rotation about a Line. Symmetry.

This case is frequently encountered in the proofs of the theorems of both plane and solid geometry. A simple illustration is afforded by the following exercise:

The diameter perpendicular to the chord of a circle bisects the chord and the subtended arc.



Draw the diameter A A′ perpendicular to the chord P P′. Since A A′ is the diameter of the circle, then the semi-circle A P A′, when rotated about A A′, coincides with the semi-circle A P′ A′. Hence P, a point on the right hand semi-circle, falls somewhere upon the left hand semi-circle.

Now, since the angles C B P, C B P′ are right-angles, then P falls somewhere on the line B P′. But as P falls on both semi-circle and straight line, it must fall at their intersection P′; and hence B P = B P′. Hence it follows, from the definition of a circle, that the arcs P A, P A′ fall upon and coincide throughout with the arcs P′ A, P′ A′. This is an instructive illustration of the intersection of loci.

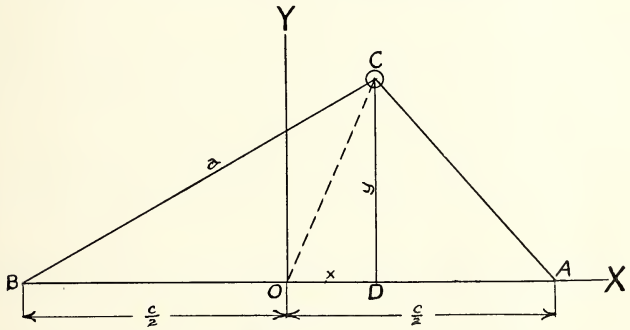
9. *The Algebraic Method.*

The best algebras written today enter at an early stage into a discussion of rectangular co-ordinates and the employment of two fixed axes at right angles as frame of reference for the position of points, lines, and other configurations. This method is the most generalized of all methods, being identical with analytic geometry. All the way through Euclidian geometry, algebraic equations, some of them of considerable complexity, are freely employed. Many problems of loci are thus solved with an elegance and rapidity attainable through the employment of no other method. It is assumed that the only instruments which may be employed in Euclidian geometry are the straight edge and compass. Investigation has established the fact, which may be stated here, that only when the equations of conditions regarding a given problem have their roots expressible as rational

quantities or square roots, can the problem in general be solved solely by the employment of straight edge and compass.

A problem already treated by another method serves excellently here to illustrate the algebraic method.

If the sides of a triangle are connected by the relation $a^2 + b^2 = c^2$, *then the triangle is right-angled at C.*



Choose as axes the base C of the triangle and the perpendicular bisector of the base.

Now by hypothesis, $a^2 + b^2 = c^2$
From the triangle B C D, $a^2 = (\frac{c}{2} + x)^2 + y^2$
From the triangle A C D, $b^2 = (\frac{c}{2} - x)^2 + y^2$

Hence $c^2 = \frac{c^2}{2} + 2x^2 + 2y^2$, *or* $x^2 + y^2 = \frac{c}{4}^2$
But $x^2 + y^2 = \overline{OC}^2$ and therefore $OC = \frac{c}{2}$

Hence A and B are the extremities of the diameter of a circle, centre O, passing through C. Therefore the angle C is a right angle, being measured by one-half a semi-circumference.

This problem may be stated under a slightly different form:

Find the locus of a point which moves so that the sum of the squares of its distances from two fixed points A, B $(AB = c)$ *is constant* $(= c^2)$.

10. *The Method of Similarity.*

Many problems arise in practice, especially problems of construction, which depend upon the principles underlying the similarity of figures. By the employment of a figure similar to the required one, the relations sought may be discovered.

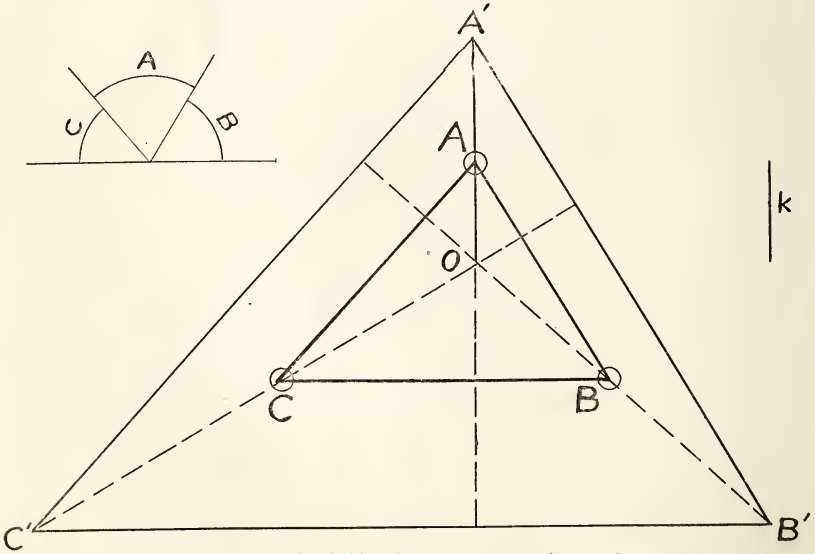
Thus, if a required figure has given at the outset but one line

segment, we may temporarily disregard its actual length, and begin by constructing a figure similar to the given one. By analyzing this figure, the desired relations may be discovered; and upon the basis of these relations, the desired figure may be constructed with the use of the line-segment of given length.

This method is capable of extension to other problems of similar character, such as problems in which two line-segments are given. Thus, in the last case, begin by constructing a figure in which the two line-segments have lengths, separately arbitrary, but in the requisite ratio; and then construct the required figure, changing the dimensions in the given proportion.

A simple illustration, which nevertheless embodies the kernel of the similarity principle, is the following exercise in construction:

To construct a triangle, having given the angles and upper segment of the triangle-altitude on the base a.



Construct a triangle A′ B′ C′ of any convenient size with angles A′, B′, C′ equal respectively to the given angles A, B, C of the required triangle. Draw the three altitudes of the triangle A′ B′ C′; and on O A′ lay off O A equal to K, the given line-segment. The triangle with A as vertex and O A as upper-segment of altitude on side *a* may now obviously be constructed by

drawing through A lines parallel respectively to A′ B′, A′ C′, cutting O B′ and O C′ in B and C, and joining the points B and C. The correctness of the construction may be shown by the comparison of several similar triangles. Thus O A : O A′ = A B : A′ B′ = O B : O B′. Also O A : O A′ = A C : A′ C′ = O C : O C′. Hence O B : O B′ = O C : O C′, and so B C is parallel to B′ C′.

All problems of this nature depend upon the fundamental principle: *In similar figures, the corresponding sides are in proportion.* By resolving proportional figures of any size and shape into a succession of similar triangles, similarly placed, it is clear that in general problems of this type depend upon the constructon of a triangle similar to a given triangle. Various theorems dealing with the similarity of triangles are available; and these must be invoked at will by the pupil to enable him to solve the problems which may arise.

In conclusion, it may safely be assumed that the methods outlined and illustrated above by no means exhaust the list of definite methods employed in the demonstration of theorems, the solution of exercises, and the construction of required figures in Euclidian geometry. The Method of Limits, for example, is so frequently employed anl so explicitly set forth in geometries now in common use, that there is no need for its further exposition in this place. It is recommended to the interested reader, especially the teacher of geometry, to attempt to outline a method of attack, of a reasonably general application, in addition to the methods set forth in the present monograph.

VI. PROCEDURE IN ATTACKING GEOMETRICAL PROBLEMS

All problems, of whatever kind, even those the solution of which is readily surmised, should be treated in a logical and systematic manner. The steps, in the order of time and logic, are as follows:

1. The problem should be stated explicitly and succinctly, yet with sufficient fulness to obviate all doubt or uncertainty regarding its nature.

2. The *data,* that is, the given or assumed entities and re-

lationships, should likewise be stated in a simple form and placed together in a paragraph.

3. The required relations to be established or constructions to be effected should next be set down with accuracy and lucidity.

4. The figure should be drawn at any time most convenient, after the original enunciation of the problem. Since analysis is the basic instrument of logic for the solution of geometrical problems, it is desirable that a figure be drawn expressing, with as much accuracy as the given data afford, the relationships sought. Particular pains should be taken to draw the figure as accurately as the instruments at hand and the data in question afford. It is indeed often the case that the tentative figure thus neatly constructed suggests the procedure of the analysis, and indicates properties and relationships not mentioned in the enunciation of the problem.

5. At this point, the investigator should pause to examine the problem; and to marshal to his support all the heavy and light artillery. In the enunciation of the things given and things required, it is desirable for the pupil to follow Pascal's advice and "substitute the definition for the name of the thing defined." In other words, give definiteness at once to the pupil's ideas by attaching the meaning in the statement to specific parts of the figure, indicated by letters, numerals, or other symbols. It is suggested that, in the earlier stages of his geometrical study, the pupil endeavor to recall, not in a haphazard manner but in an orderly array, the theorems and even fundamental exercises already proven, one or more of which, at least, must certainly be resorted to in order to establish the desired properties or relations. Thus, if two triangles are to be proven equal, all the theorems establishing the equality of a pair of triangles may be listed. Following the listing of this irreducible minimum of possibly applicable theorems, the pupil may then attempt to apply in turn each theorem to the particular problem in question. By eliminating the theorems readily recognized as not applicable, the field of investigation is quickly narrowed to a reasonable range; and the problem may then be more intensively studied with reference to the remaining theorems which clearly do apply to the problem.

This method, because of its mechanical character and needlessly lengthy procedure, should be discarded as soon as the student has acquired a working knowledge of a considerable number of the fundamental theorems. Its continued use after a certain stage of proficiency has been attained by the student, should be discouraged; since its tendency is to scatter, rather than to focus, the attention upon the essential features of the geometrical problem to be solved. As soon as the student has attained the desired proficiency, he should then be urged to concentrate his attention upon the essential geometric features and conditions of the problem; and, through such concentration, thereby be enabled to identify the theorem or theorems both necessary and sufficient for logical procedure towards the proof. The rejection of theorems which are not applicable thus becomes an unconscious, almost automatic process; the student himself soon ceases to be aware of it. The method suggested for the student's use in the earlier stages of his study is the method of indirection, and reaches its goal by the longest route that can properly be called methodical. The method ultimately to be employed, as outlined above, is the direct method—marked by intensiveness and immediacy.

6. In many problems of average or more than average difficulty, it is often the case that none of the known theorems pertain directly to the figure as drawn. In such cases, auxiliary lines and configurations, i. e., straight lines and curved lines, either arcs or entire circles, must be drawn, thus setting up a group of new relationships between the various parts of the figure and permitting the application to the enlarged or more elaborate figure of theorems and principles not hitherto pertinent. Great ingenuity is often required for the drawing of the requisite auxiliary lines; and a knowledge of the various forms of symmetry, of the fundamental principles of translation and rotation, and of the basic ideas underlying similarity will prove of the greatest value. Method alone will not provide for or supply the multiplicity of possible figures and the rich functioning of the human intelligence. Auxiliary lines, of various sorts and in different positions, may be drawn, affording many separate solutions, distinct from each other, of the same problem. Furthermore, the particular ability and maturity in logical analysis of the in-

dividual student are incalculable factors which nevertheless are of dominating importance.

7. Once the theorem or theorems applicable to the given problems have been disengaged and their relations to the problem definitely determined, a careful analysis of the problem should be carried through in the fashion set forth in the early part of the present monograph. If a direct proof cannot be effected, the attempt should be made to establish the theorem by the indirect method, in the fashion already explained.

8. Following the analytical exposition of the problem, the procedure should be reversed; and a synthetic proof, in succinct form, should be carried through. If the problem involves the construction of loci, in other words, is what is known as a problem of construction, the analysis preceding the synthetic proof should provide for all possible solutions, and furnish machinery for the accurate construction of all the loci required.

VII. BIBLIOGRAPHICAL NOTE

The most scientific analysis, but scarcely adapted for the use of beginners, of the question of the solution of geometrical problems is the work of Julius Petersen: *Methods and Theories for the Solution of Geometric Problems of Construction,* which has been published in English, German, and French (English translation, Low & Co., London. 1879; German translation, Copenhagen, 1879; French translation, Gauthier-Villars, Paris, 1880). A similar work is Alexandroff's *Problems of Elementary Geometry,* which has appeared in both French (Paris, 1899) and German (Leipzig, 1903). In English, two works of mediocre interest and value, being scarcely more than pamphlets, are E. S. Loomis's *Original Investigation, or How to Attack an Exercise in Geometry* (Ginn & Co., Boston. 1901), and J. C. Stone's *Method in Geometry* (D. C. Heath & Co., 1904). Works of especial interest to teachers in connection with the subjects dealt with in the present monograph are A. Shultze's *The Teaching of Mathematics in Secondary Schools* (The Macmillan Co., 1912), D. E. Smith's *The Teaching of Geometry* (Ginn & Co., 1911), and J. W. A. Young's *The Teaching of Mathematics* (Longmans, Green & Co., 1907). Three geometries which might

be studied with profit by the teacher of geometry are Edwards's *Elements of Geometry* (1895), Beman and Smith's *Plane and Solid Geometry* (1895), and Halsted's *Rational Geometry* (1904). To the last mentioned I am particularly indebted. With the exception of Alexandroff's work, to which I have not had access, and Edwards's *Elements of Geometry,* which was not available, all of the works cited above have been of assistance in the preparation of the present monograph. The treatment here given, however, is the consummation of long experience and constant experimentation in the teaching of geometry. It should be stated that the illustrations herein employed are the simplest, as well as the most creatively suggestive, which I could discover, to clarify and elucidate the methods of attacking a problem in geometry.